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Ringel-Hall algebra of A_∞-type*

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Abstract: The category of the finite-dimensional representations of kA_∞ was studied first, with all its indecomposable objects and their extenswere were given explicitly, the Ringel-Hall algebra $H(kA_\infty)$ was investigated for a finite field k was investigated. The main viewpoint of this investigation is to regard $H(kA_\infty)$ as the direct limit of the Ringel-Hall algebra $H(kA_n)$. In particular, a PBW-basis of $H(kA_\infty)$ was gotten. The investigation shows that $H(kA_\infty)$ coincides with its composition subalgebra.

Key words: quiver; path algebra; quantum group; Ringel-Hall algebra; direct limit

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A_∞ 型 Ringel-Hall 代数

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摘要:首先研究建立在任意域 k 上的 A_∞ 型路代数 kA_∞ 的有限维模范畴,给出了 kA_∞ 的有限维模范畴与 A_∞ 的有限子 quiver 所对应的路代数上的有限维模范畴之间的关系,特别的具体的给出了所有的不可分解有限维 kA_∞ 模,精确的刻画了不可分解模之间的模扩张;然后给定有限域 k,研究了建立在有限维 kA_∞ 模范畴上的 Ringel-Hall 代数 $H(kA_\infty)$. 证明了 $H(kA_\infty)$ 恰好是当 n 趋向 ∞ 时 $H(kA_n)$ 的正向极限,特别的找到了 $H(kA_\infty)$ 的一个 PBW 基,并且证明 $H(kA_\infty)$ 恰好与它的合成子代数相符合.

关键词:箭图;路代数;量子群;Ringel-Hall 代数;正向极限

0 Introduction

Given a finite quiver Q without oriented cycles, one has the corresponding symmetric Cartan matrix, and then the corresponding Kac-Moody algebra and its quantized enveloping algebra $U=U(Q)$. On the other hand, one has the Ringel-Hall algebra $H(kQ)$ of the path algebra kQ over a finite field k. The most important progress of the study of quantum groups in the last decade is that, as invented by Ringel[1,2], Green[3] and Lusztig[4], the positive part U^+ of U is isomorphic to, in a canonical way, the twisted and generic version of Ringel's composition subalgebra $C(kQ)$ of $H(kQ)$. This isomorphism has been extended to U by Xiao in Ref. [5] by using the Drinfeld double,

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also see Ref. [6](Deng and Xiao). This provides a framework of the Ringel-Hall algebra's approach to quantum groups.

The natural question then is whether or not this approach also works for infinite quivers. The first step towards this effort is to look at the quiver of A_∞ type and the corresponding quantum group $U_q(sl_\infty)$. This is the aim of the present paper.

In order to study the quantum group of type sl_∞ via Ringel-Hall algebra of A_∞ type, first, we need to deal with finite-dimensional representations of the infinite-dimensional path algebra kA_∞ over any field k. Notice that kA_∞ is a infinite-dimensional algebra without identity element and there exist no projective objects in its category of finite-dimensional modules. However, a finite-dimensional kA_∞-module can be viewed as a module of a path algebra of type A_n for some $n \in \mathbb{N}$; and the Ringel-Hall algebra $H(kA_\infty)$ can be viewed as the direct limit of the Ringel-Hall algebra $H(kA_n)$.

In section 1 we studied the category of the finite-dimensional representations of kA_∞ by determining all its indecomposable objects and their extensions explicitly. In section 2 we investigated the Ringel-Hall algebra $H(kA_\infty)$ by calculating the product of two modules of $H(kA_\infty)$ inside $H(kA_n)$ for suitable $n \in \mathbb{N}$. In particular, we got a PBW-basis of $H(kA_\infty)$, and showed that $H(kA_\infty)$ is a composition algebra.

In this paper, $\mathbb{N}$ denotes the set of positive integers. All modules are finite-dimensional left modules. Denote by $|X|$ the cardinality of a set X.

1 Finite-dimensional representations of path algebras of type A_∞

A quiver Q consists of $Q = (Q_0, Q_1, h, t)$, where Q_0, Q_1 are two sets, which are respectively called the set of vertices and the sets of arrows of Q, and h, t are two maps from Q_1 to Q_0 for which $h(\alpha)$ and $t(\alpha)$ are respectively called the head and the tail of $\alpha \in Q_1$. A path p in Q of length l means a sequence of arrows $p = \alpha_l \cdots \alpha_1$ with $t(\alpha_i) = h(\alpha_{i+1})$ for $1 \leqslant i \leqslant l-1$. Set $h(p) = h(\alpha_1), t(p) = t(\alpha_l)$ and $l(p) = l$, which are called the head, the tail and the length of p respectively. Regard a vertex $i \in Q_0$ as a path of length 0 and denote it by e_i.

For any field k and any quiver Q, let kQ be the k-space with basis the set of all finite length paths in Q. For any two paths $p = \alpha_m \cdots \alpha_1$ and $q = \beta_n \cdots \beta_1$ in Q, define the multiplication

$$qp = \begin{cases} \beta_n \cdots \beta_1 \alpha_m \cdots \alpha_1, & \text{if } t(p) = h(q), \\ 0, & \text{otherwise.} \end{cases}$$

Then kQ becomes a k-algebra, which is called the path algebra of Q. A representation (V, f) of a quiver Q over a field k is given by a vector space V_i for each $i \in Q_0$ and a k-linear maps $f_\alpha : V_{h(\alpha)} \to V_{t(\alpha)}$ for each arrow $\alpha \in Q_1$. We say the representation (V, f) is finite dimensional over k if $\bigoplus_{i \in Q_0} V_i$ is. It is well-known that the category of finite dimensional representations of a finite quiver Q over a field k is equivalent to the category of finite-dimensional kQ-modules. For $n \in \mathbb{N}$, consider the path algebra kA_n of the following quiver A_n:

$$\overset{1}{\bullet} \xrightarrow{\alpha_1} \overset{1}{\bullet} \xrightarrow{\alpha_2} \cdots \overset{1}{\bullet} \xrightarrow{\alpha_{n-1}} \overset{n}{\bullet}$$

Remark 1.1 It is well-known that the quiver A_n is of finite representation type, that is, there are only finitely many non-isomorphic indecomposable kA_n-modules. All non-isomorphic indecomposable representations of kA_n are given as follows,

$$\overset{(1)}{0} \longrightarrow \cdots \longrightarrow \overset{(i)}{k} \xrightarrow{1_{id}} \cdots \xrightarrow{1_{id}} \overset{(i+s-1)}{k} \longrightarrow 0 \cdots \longrightarrow \overset{(n)}{0},$$

where $1 \leqslant i \leqslant n$, and $1 \leqslant s \leqslant n-i+1, i, s \in \mathbb{N}, 1_{id}$ is the identity map. For general representation theory of finite-dimensional hereditary algebras please refer to Refs. [7] and [8].

Denote by kA_∞ the path algebra of the quiver A_∞

$$\overset{1}{\bullet} \xrightarrow{\alpha_1} \overset{1}{\bullet} \xrightarrow{\alpha_2} \cdots \overset{1}{\bullet} \xrightarrow{\alpha_{n-1}} \overset{n}{\bullet} \cdots$$

Remark 1.2 Note that kA_∞ has no identity element according to the definition of path algebras. It is easy to check that kA_∞ is a infinite dimensional k-algebra. Throughout the paper a module E over kA_∞ is always supposed to satisfy the condition $kA_\infty E = E$, which is equivalent to $E=\bigoplus_{i\in\mathbb{N}} e_iE$.

Denote the category of finite dimensional kA_∞-modules (resp. kA_m-modules) by kA_∞-mod (resp. by kA_m-mod).

Lemma 1.1 For any kA_∞-module E there exist unique integers $m\in\mathbb{N}$ such that $e_mE\neq 0$, and $e_jE=0$ for any $j>m, j\in\mathbb{N}$. Moreover E can be naturally viewed as a unitary kA_m-module.

Proof Since $E=\bigoplus_{i\in\mathbb{N}}e_iE$ and E is finite dimensional, there are only finitely many i such that $e_iE\neq 0$. Thus one can take m to be the largest integer i satisfying $e_iE\neq 0$. □

Remark 1.3 Lemma 1.1 provides a way to associate a given kA_∞-module E with a positive integer m in a unique way such that $E=kA_mE, e_mE\neq 0$. If it's the case, we say that E is of m-type.

Lemma 1.2 If E is a kA_m-module, then E is also a kA_∞-module (resp. kA_n-module if $n\geqslant m$) such that $e_iE=0$ for any $i>m, i\in\mathbb{N}$ (resp. for $m<i\leqslant n, i\in\mathbb{N}$).

Proof Let $\rho: kA_m\to End_k(E)$ be the k-algebra morphism giving E the kA_m module structure. It is easy to check that $kA_m\cong kA_\infty/\mathcal{J}$, where $\mathcal{J}$ is the ideal of kA_∞ generated by $\{\alpha_i, e_j \mid i\notin[1,m), j\notin[1,m], i,j\in\mathbb{N}\}$. Let $\pi: kA_\infty \twoheadrightarrow kA_m$ be the canonical k-algebra epimorphism. Then we get a k-algebra morphism $\rho\pi: kA_\infty\to End_k(E)$. Thus E also becomes a kA_∞-module. Concretely, for any $u\in E$ and any path p in A_∞ we have

$$pu=\begin{cases}\rho(p)(u), & \text{if } p \text{ is a path in } A_m;\\ 0, & \text{otherwise.}\end{cases}$$

□

Lemma 1.3 Let E, F be kA_∞-modules and $f\in Hom_{kA_\infty}(E,F)$. Then there exists $m\in\mathbb{N}$, such that $f(pE)=0$ for any path in A_∞ with $p\notin A_m$ and $f\in Hom_{kA_m}(E,F)$.

Proof Suppose that E is of r-type and F is of u-type. Let $m=\max\{r,u\}$. Then E and F can be viewed as kA_m-modules by lemma 1.2. For any path p in A_∞ with $p\notin A_m$ one has $pE=0$, so $f(pE)=0$. Since $f\in Hom_{kA_\infty}(E,F)$, it certainly holds $f\in Hom_{kA_m}(E,F)$. □

Theorem 1.1 kA_m-mod can be regarded as a fully faithful and extension closed subcategory of kA_∞-mod (resp. kA_n-mod if $m<n$).

Proof Let M, $N\in kA_m$-mod, then M, N can be viewed as kA_∞ modules by lemma 1.2. For $f\in Hom_{kA_m}(M,N)$, one has $f\in Hom_{kA_\infty}(M,N)$ since for $m\in M$, p a path in A_∞,

$$f(pm)=\begin{cases}pf(m), & \text{if } p \text{ is a path in } A_m;\\ 0=pf(m), & \text{otherwise.}\end{cases}$$

Thus one can define a map $\phi: Hom_{kA_m}(M,N)\to Hom_{kA_\infty}(M,N)$ via $\phi(f)=f$ for $f\in Hom_{kA_m}(M,N)$. Conversly if $f\in Hom_{kA_\infty}(M,N)$, certainly $f\in Hom_{kA_m}(M,N)$. Thus one can define a map $\varphi: Hom_{kA_\infty}(M,N)\to Hom_{kA_m}(M,N)$ via $\phi(f)=f$ for $f\in Hom_{kA_\infty}(M,N)$. It is easy to check that $\varphi\phi=id$ and $\phi\varphi=id$, thus $Hom_{kA_m}(M,N)=Hom_{kA_\infty}(M,N)$. let $0\to M\to E\to N\to 0$ be a short exact sequence in kA_∞-mod, then $e_jE=0$ for any $j>m$, $j\in\mathbb{N}$ since $e_jM=e_jN=0$. Then this is just a short exact sequence in kA_m-mod.

Theorem 1.2 If E is a kA_∞-module of m-type, then E is indecomposable if and only if E is indecomposable as a kA_m-module. Moreover, all finite-dimensional indecomposable kA_∞-representations are given by

$$\cdots 0\to \overset{(a)}{k}\xrightarrow{1_{id}} k\xrightarrow{1_{id}}\cdots\xrightarrow{1_{id}}\overset{(b)}{k}\to 0\cdots$$

with $a\leqslant b, a,b\in\mathbb{N}$.

Proof it is easy to prove by lemma1.1, lemma 1.2, theorem 1.1 and remark 1.1. □

For convenience, we denote the indecomposable kA_∞-module corresponding to the representation

$$\cdots 0\to \overset{(a)}{k}\xrightarrow{1_{id}} k\xrightarrow{1_{id}}\cdots\xrightarrow{1_{id}}\overset{(b)}{k}\to 0\cdots$$

by ind_b^a with $a\leqslant b, a,b\in\mathbb{N}$.

Define the length of ind_b^a to be $b-a+1$, denote it by $l(\mathrm{ind}_b^a)$. For any kA_∞-module E, if it is isomorphic to $\bigoplus_{i=1}{}^lE_i$, define $l(E)$, the length of E, to be $\sum_{i=1}^{l}l(E_i)$, where E_i are indecomposable kA_∞-modules.

Let $\{S_i \mid i\in\mathbb{N}\}$ be all the pairwise non-isomorphic simple kA_∞-modules. For $E\in$

kA_∞-mod, denote by $\underline{\dim}\, E=((\underline{\dim}\, E)_{e_i})_{i\in\mathbb{N}}$ the dimension vector of E, i. e. $(\underline{\dim}\, E)_{e_i}$ is the Jordan-Hölder multiplicity of S_i in E. Given kA_∞-modules A and B, consider the set $\hat{E}_{kA_\infty}(A,B)$ consisting of all such short exact sequences of kA_∞-modules: $0\to B\to E\to A\to 0$. Two short exact sequences $0\to B\to E_1\to A\to 0$ and $0\to B\to E_2\to A\to 0$ in $\hat{E}_{kA_\infty}(A,B)$ are said to be equivalent if there is a homomorphism of kA_∞-modules $\alpha: E_1\to E_2$ such that the diagram commutes. It is easy to check that the relation defined above is an equivalent one. Denote the set of equivalence classes of $\hat{E}_{kA_\infty}(A,B)$ by $E_{kA_\infty}(A,B)$. For any kA_m-modules E,F, $\hat{E}_{kA_m}(E,F)$ and $E_{kA_m}(E,F)$ can be defined in a similar way.

$$\begin{array}{ccccccccc} 0 & \to & B & \to & E_1 & \to & A & \to & 0 \\ & & \| & & \downarrow{\scriptstyle\alpha} & & \| & & \\ 0 & \to & B & \to & E_2 & \to & A & \to & 0 \end{array} \qquad (1)$$

Lemma 1. 4 Let M, N be kA_∞-modules of r-type and u-type respectively. If there is a short exact sequence in kA_∞-mod: $0\to M\to E\to N\to 0$, then E is of m-type where $m=\max\{r,u\}$. In particular the short exact sequence can be viewed as a short exact sequence in kA_m-mod.

Proof Since $\underline{\dim}\, E=\underline{\dim}\, M+\underline{\dim}\, N$. □

Theorem 1. 3 Let M, N be kA_∞-modules of r-type and u-type respectively. Let $m=\max\{r,u\}$. Then there is a bijection of sets $\hat{\eta}$: $E_{kA_\infty}(M,N)\simeq Ext^1_{kA_m}(M,N)$.

Proof On the one hand, any such short exact sequence $0\to N\to E\to M\to 0$ in kA_∞-mod is also a short exact sequence in kA_m-mod by lemma 1. 4. Moreover the above diagram (1) as kA_m-modules is commutative if as kA_∞-modules the diagram is.

On the other hand, M and N can be viewed as kA_m-modules, and any short exact sequence of kA_m-modules $0\to N\to E\to M\to 0$ can be viewed as a short exact sequence of kA_∞-modules by lemma 1. 2 and theorem 1. 1. Also as kA_∞-modules the above diagram (1) is commutative if as kA_m-modules the diagram is.

So the set $\hat{E}_{kA_\infty}(M,N)$ is isomorphic to $\hat{E}_{kA_m}(M,N)$. The set $E_{kA_\infty}(M,N)$ is isomorphic to $E_{kA_m}(M,N)$. It is well known that there is a bijection of sets

$$\eta: E_{kA_m}(M,N)\simeq Ext^1_{kA_m}(M,N).$$

Thus as set $E_{kA_\infty}(M,N)$ is isomorphic to $Ext^1_{kA_m}(M,N)$. □

Corollary 1. 1 If A, B are kA_∞-modules, then the set $E_{kA_\infty}(A,B)$ has a natural abelian group structure.

Proof Since there is an isomorphism of sets $\hat{\eta}: E_{kA_\infty}(A,B)\simeq Ext^1_{kA_m}(A,B)$, for some $m\in\mathbb{N}$ by theorem 1. 3, and since $Ext^1_{kA_m}(A,B)$ carries a natural abelian group structure. □

Corollary 1. 2 For any kA_∞-module E, $E_{kA_\infty}(E,\mathrm{ind}^1_m)$ consists of only one element for $m\in\mathbb{N}$.

Proof Assume that E is of r-type for some $r\in\mathbb{N}$. Let $n=\max\{r,m\}$. Then the kA_∞-module ind^1_m can be viewed as a kA_n-module, and so can the kA_∞-module E. Furthermore ind^1_m is a injective kA_n-module, so $Ext_{kA_n}{}^1(E,\mathrm{ind}^1_m)=0$. Hence $E_{kA_\infty}(E,\mathrm{ind}^1_m)$ has only one element by theorem 1. 3. □

2 The Ringel-Hall algebra $H(kA_\infty)$

Let k be a finite field with $|k|=q<\infty$. In the following we will only concern about the category of finite left kA_∞-modules, which is denoted by kA_∞-fin. Here a finite module means that it only contains finitely many elements. Because finite kA_∞-modules are exactly finite-dimensional kA_∞-modules, it follows that kA_∞-fin=kA_∞-mod.

Lemma 2. 1 $E_{kA_\infty}(M,N)$ is a finite set for $M,N\in kA_\infty$-mod.

Proof It follows from theorem 1. 3. □

By lemma 2. 1 we know that kA_∞ is a finitary ring.

Let M, N_1, …, $N_t\in kA_\infty$-mod. Denote by $g^M_{N_1,\cdots,N_t}$ the number of filtrations

$$M=M_0\supseteq M_1\supseteq\cdots\supseteq M_t=0$$

of M such that $M_{i-1}/M_i\cong N_i$ for $1\leqslant i\leqslant t$.

Remark 2. 1 Assume that M is a kA_∞-module of m-type. Then any submodule and quotient module of M can also be viewed as kA_m-modules.

As kA_∞-modules any filtration $M=M_0\supseteq M_1\supseteq\cdots\supseteq M_t=0$ of M such that $M_{i-1}/M_i\cong N_i$ for $1\leqslant i\leqslant t$ can be viewed as a filtration of kA_m-module M. On the other hand any filtration $M=M_0\supseteq M_1\supseteq\cdots\supseteq M_t=0$ of kA_m-module M such that $M_{i-1}/M_i\cong N_i$ for $1\leqslant i\leqslant t$ can be viewed as a filtration of kA_∞-module M by lemma 1.2. So to compute $g^M_{N_1,\cdots,N_t}$ we just need to determine the number of filtrations $M=M_0\supseteq M_1\supseteq\cdots\supseteq M_t=0$ of kA_m-module M such that $M_{i-1}/M_i\cong N_i$ for $1\leqslant i\leqslant t$.

For any $M\in kA_\infty$-mod denote by $[M]$ the isomorphic class of M. The following lemma comes from two different ways to count the number of filtration $M\supseteq M_1\supseteq M_2$ of M such that $M/M_1\cong N_1$, $M_1/M_2\cong N_2$, $M_2\cong N_3$.

Lemma 2.2 For M, N_1, $N_2\in kA_\infty$-mod, we have

$$\sum_{[L]} g^L_{N_1,N_2}g^M_{L,N_3}=\sum_{[L]} g^M_{N_1,L}g^L_{N_2,N_3}=g^M_{N_1,N_2,N_3}.$$

Let $H(kA_\infty)$ denote the $\mathbb{Q}$-space with basis $\{[M]\mid M\in kA_\infty\text{-mod}\}$. Consider the multiplication on $H(kA_\infty)$: $[N_1]\circ[N_2]:=\sum_{[M]}g^M_{N_1,N_2}[M]$. Note that by lemma 2.1 the sum is a finite sum.

Lemma 2.3 With the above construction, $H(kA_\infty)$ is an associative $\mathbb{Q}$-algebra with identity $[0]$, which is called the Ringel-Hall algebra of kA_∞.

Proof Note that the associativity of the multiplication defined above follows from the fact that the coefficients of $[M]$ in $([N_1]\circ[N_2])\circ[N_3]$ and $[N_1]\circ([N_2]\circ[N_3])$ are given by $\sum_{[L]}g^L_{N_1,N_2}g^M_{L,N_3}$ and $\sum_{[L]}g^M_{N_1,L}g^L_{N_2,N_3}$ respectively, and hence they are equal by lemma 2.2. □

Ringel-Hall algebra $H(kA_m)$ for $m\in\mathbb{N}$ can be defined similarly. Here we recall an important formula to compute coefficients. Let M,N be kA_m-modules. Denote by a_X the order of automorphism group $Aut_{kA_m}(X)$ for any kA_m-module X. Let $Ext_{kA^1_m}(M,N)_L$ be the set of extension classes in $Ext_{kA^1_m}(M,N)$ with middle term L. Then

$$g^L_{M,N}=\frac{a_L\,|Ext^1_{kA_m}(M,N)_L|}{a_Ma_N\,|Hom_{kA_m}(M,N)|}.\tag{2}$$

See Ref. [9].

Suppose M, N are kA_∞-modules of r-type and u-type respectively. Set $m=\max\{r,u\}$, then by remark 2.1 and formula (2) we have

$$g^L_{M,N}=\frac{a_L\,|Ext^1_{kA_m}(M,N)_L|}{a_Ma_N\,|Hom_{kA_m}(M,N)|}.\tag{3}$$

Proposition 2.1 Let M_1, M_2, $\cdots$, M_t be kA_∞-modules of m_i-type for $i=1,2,\cdots,t$ respectively. Set $n\in\mathbb{N}$, $n\geqslant m=\max\{m_i\mid i=1,\cdots,t\}$. If in $H(kA_n)$

$$[M_1]\circ\cdots\circ[M_t]=\sum_{[L]}g^L_{M_1,\cdots,M_t}[L],$$

when M_i are viewed as kA_n-modules, then in $H(kA_\infty)$ there holds

$$[M_1]\circ\cdots\circ[M_t]=\sum_{[L]}g^L_{M_1,\cdots,M_t}[L],$$

where all L's in the last summation are viewed as kA_∞-modules.

Proof Firstly if in $H(kA_\infty)$

$$[M_1]\circ\cdots\circ[M_t]=\sum_{[L]}g^L_{M_1,\cdots,M_t}[L],$$

then we claim that any kA_∞-module L with $JP2g^L_{M_1,\cdots,M_t}\neq 0$ is of m-type. To verify it just use lemma 1.4 repeatedly. By lemma 1.2 we can view L as a kA_n-module.

Secondly when $M_1,M_2,\cdots,M_t$ are viewed as kA_n-modules, if in $H(kA_n)$

$$[M_1]\circ\cdots\circ[M_t]=\sum_{[L]}g^L_{M_1,\cdots,M_t}[L],$$

then kA_n-modules L can be viewed as kA_∞-modules by lemma 1.2. And it is easy to check that such kA_∞-module L must be of m-type. Moreover, by remark 2.1 corresponding coefficients $g^L_{M_1,\cdots,M_t}$ are equal. This completes the proof. □

Theorem 2.1 $H(kA_m)$ can be naturally viewed as a subalgebra of $H(kA_\infty)$ by sending $[L]$ in $H(kA_m)$ to corresponding $[L]$ in $H(kA_\infty)$ for $m\in\mathbb{N}$.

Proof It follows from proposition 2.1 immediately. □

Similarly, we can prove that $H(kA_m)$ can be naturally viewed as a subalgebra of $H(kA_n)$ if $m\leqslant n$. Given a finite $\mathbb{Z}$-module M, denote its length by $l_{\mathbb{Z}}(M)$.

Lemma 2.4 Let R be a finitary ring, and let

$L \in R$-fin have a filtration with factors $M_1, \cdots, M_t$. Then $l_{\mathbb{Z}}(Ext_R{}^1(L,L)) \leqslant l_{\mathbb{Z}}(Ext_R{}^1(M_1 \oplus \cdots \oplus M_t, M_1 \oplus \cdots \oplus M_t))$. Moreover, the equality holds if and only if $L \cong M_1 \oplus \cdots \oplus M_t$.

Proof See Ref. [10]. □

Denote by ind-kA_∞ the full subcategory of kA_∞-mod consisting of indecomposable kA_∞-modules.

Corollary 2. 1 $H(kA_\infty)$ is generated by all $[M]$ with $M \in$ ind-kA_∞.

Proof Let $M = M_1 \oplus \cdots \oplus M_t$ with M_i indecomposable where $i = 1, \cdots, t$. Assume that M is a of m-type, then M_i can also be viewed as kA_m-modules. Use induction on $l_{\mathbb{Z}}(E_{kA_\infty}(M,M))$. If $|E_{kA_\infty}(M,M)| = 1$, then $[M] = c[M_1] \circ \cdots \circ [M_t]$ for some $0 \neq c \in \mathbb{Q}$. If $|E_{kA_\infty}(M,M)| \neq 1$, then

$$[M_1] \circ \cdots \circ [M_t] = c[M] + \sum_{[L] \neq [M]} c_L[L],$$

where in the summation L runs over all modules such that $c_L \neq 0$. Since kA_∞ is a finitary ring, we have

$$l_{\mathbb{Z}}(E_{kA_\infty}(L,L)) < l_{\mathbb{Z}}(E_{kA_\infty}(M,M)),$$

by lemma 2. 4. Then the assertion follows by induction. □

One can introduce a total ordering on the set of isoclass of indecomposable kA_∞-modules, i. e. , the set $\{\mathrm{ind}_b^a\}_{a,b \in \mathbb{N}}$. For each pair of indecomposable kA_∞-modules ind_b^a and ind_d^c, $\mathrm{ind}_b^a < \mathrm{ind}_d^c$ if $a < c$ or $a = c$ and $b < d$. That is

$$\mathrm{ind}_1^1 < \mathrm{ind}_2^1 < \cdots < \mathrm{ind}_2^2 < \mathrm{ind}_3^2 < \cdots \cdots.$$

Proposition 2. 2 Let $E < F$ be a pair of indecomposable kA_∞-modules. Then $Hom_{kA_\infty}(E, F) = 0$ and $|E_{kA_\infty}(F,E)| = 1$.

Proof Denote by S_i the simple kA_∞-module corresponding to the i-th vertex of the quiver A_∞, $i \in \mathbb{N}$. Let $E = \mathrm{ind}_b^a, F = \mathrm{ind}_d^c$ with $a \leqslant b, c \leqslant d$, $a,b,c,d \in \mathbb{N}$, then there are two cases arising, either $a < c$ or $a = c$ and $b < d$.

If $a < c$, then any quotient module of ind_b^a contains simple S_a as its top while no submodule of ind_d^c has S_a as its factor, so we must have $Hom_{kA_\infty}(\mathrm{ind}_b^a, \mathrm{ind}_d^c) = 0$. If $a = c$ and $b < d$, since ind_d^c is a uniserial module and ind_b^a is a proper quotient module of ind_d^c, and any quotient module of ind_b^a does not have S_d as its socle, we have $Hom_{kA_\infty}(\mathrm{ind}_b^a, \mathrm{ind}_d^c) = 0$.

Recall that for finite-dimensional kA_n-modules X and Y with $n \in \mathbb{N}$, there holds $DExt^1_{kA_n}(X,Y) \cong Hom_{kA_n}(\tau^{-1}Y, X)$ where τ is the Auslander-Reiten translation, $D = Hom_k(-, k)$. It is easy to get $\tau^{-1}\mathrm{ind}_b^a = \mathrm{ind}_{b-1}^{a-1}$ if $a > 1$ when we view ind_b^a as a kA_b-module. Let $f = \max\{b,d\}$, then ind_b^a and ind_d^c can all be viewed as kA_f-modules. If $a = 1$, then $Ext_{kA_f}{}^1(\mathrm{ind}_d^c, \mathrm{ind}_b^1) = 0$ since ind_b^1 is an injective kA_f-module. If $a > 1$, then $a - 1 < c$ since $a \leqslant c$. Therefore $Hom_{kA_f}(\tau^{-1}\mathrm{ind}_b^a, \mathrm{ind}_d^c) = Hom_{kA_f}(\mathrm{ind}_{b-1}^{a-1}, \mathrm{ind}_d^c) = 0$ by the similar reason discussed above. Thus we have $Ext^1_{kA_f}(\mathrm{ind}_d^c, \mathrm{ind}_b^a) = 0$. Hence $|E_{kA_\infty}(\mathrm{ind}_d^c, \mathrm{ind}_b^a)| = |Ext_{kA_f}{}^1(\mathrm{ind}_d^c, \mathrm{ind}_b^a)| = 1$ by theorem 1. 3. □

Let E be a kA_∞-module, $r \in \mathbb{N}$, denote by $[E]^r$ the product of r copies of $[E]$ in $H(kA_\infty)$. By using the ordering defined above, one obtains a PBW-type basis of $H(kA_\infty)$.

Lemma 2. 5 Let $0 \to M \to E \to N \to 0$ be an exact sequence of kA_∞-modules. If $Hom_{kA_\infty}(M,N) = 0$, then $g^E_{N,M} = 1$.

Proof Let $\pi: E \to N$ be an epimorphism, then since $Hom_{kA_\infty}(M,N) = 0$ one gets that $M \in ker(\pi)$. Comparing the dimensions of M, E and N, one can show that $M = ker(\pi)$. Now the lemma follows from the definition. □

Theorem 2. 2 The set

$$\Omega = \{[E_{i_1}]^{r_1} \cdots [E_{i_\alpha}]^{r_\alpha} \mid E_{i_j} \in \text{ind-}kA_\infty, \alpha \in \mathbb{Z}_{\geqslant 0}, r_j \in \mathbb{N}\}$$

gives a PBW-basis of $H(kA_\infty)$, where $E_{i_1} > E_{i_2} > \cdots > E_{i_\alpha}$ are pairwise non-isomorphic indecomposable kA_∞-modules.

Proof Let $E \in kA_\infty$-mod, then one can prove by induction that there exists t_j, $m \in \mathbb{N}$ with $j = 1, 2, \cdots, m$, such that E can be decomposed as $t_1E_{i_1} \oplus t_2E_{i_2} \cdots \oplus t_mE_{i_m}$ where E_{i_j} are pairwise non-isomorphic indecomposable kA_∞-modules and $E_{i_j} \langle E_{i_l}$ if $j \rangle l, j, l = 1, \cdots, m$ with the order defined above.

Furthermore an easy calculation by lemma 2.5 shows that $[E]=[t_1E_{i_1}]\circ[t_2E_{i_2}]\circ\cdots\circ[t_mE_{i_m}]$ in $H(kA_\infty)$. It is easy to check that if $F\in$ ind-kA_∞ then $|E_{kA_\infty}(F,F)|=1$, so in $H(kA_\infty)$ one has $[F]^r=h_r[rF]$ for some $h_r\in\mathbb{Q}$. Hence $[E]=[t_1E_{i_1}\oplus t_2E_{i_2}\cdots\oplus t_mE_{i_m}]=[t_1E_{i_1}]\circ[t_2E_{i_2}]\circ\cdots\circ[t_mE_{i_m}]=h[E_{i_1}]^{t_1}\circ\cdots\circ[E_{i_m}]^{t_m}$ for some $0\neq h\in\mathbb{Q}$. Since $[E_{i_1}]^{t_1}\circ\cdots\circ[E_{i_m}]^{t_m}=\dfrac{1}{h}[t_1E_{i_1}\oplus t_2E_{i_2}\cdots\oplus t_mE_{i_m}]$, the set Ω is linearly independent over $\mathbb{Q}$. □

For more information about PBW-basis of Ringel-Hall algebras please refer to [11].

Proposition 2.3 Multiplications of $[\mathrm{ind}_b^a]$, $[\mathrm{ind}_d^c]$ in $H(kA_\infty)$ with $a,b,c,d\in\mathbb{N}$ are given as follows.

$$[\mathrm{ind}_b^a]\circ[\mathrm{ind}_d^c]=\begin{cases}(q+1)[\mathrm{ind}_b^a\oplus\mathrm{ind}_b^a] & a=c,b=d,\\ [\mathrm{ind}_b^a\oplus\mathrm{ind}_d^c] & a>c,\ b=d,\\ q[\mathrm{ind}_b^a\oplus\mathrm{ind}_d^c] & a<c,\ b=d,\\ [\mathrm{ind}_b^a\oplus\mathrm{ind}_d^c] & b>d,\\ q[\mathrm{ind}_b^a\oplus ind_d{}^a] & c=a,\ b<d,\\ [\mathrm{ind}_b^a\oplus ind_d{}^{b+1}]+[ind_d{}^a] & c=b+1,\ b<d,\\ q[\mathrm{ind}_b^a\oplus\mathrm{ind}_d^c]+[ind_d{}^a\oplus ind_b{}^c] & c\in[a+1,b],\ b<d,\\ [\mathrm{ind}_b^a\oplus\mathrm{ind}_d^c] & c\notin[a,b+1],\ b<d.\end{cases}$$

Proof It is just to calculate directly according to (3) and properties of finite-dimensional representations of kA_∞ given in section 1. □

Let R be a finitary ring. $H(R)$ is the Ringel-Hall algebra of R. By definition the composition algebra $C(R)$ is the subalgebra of $H(R)$ generated by all isoclasses of finite simple R-modules $[S]$. For more information about $C(R)$, see Ref. [12], [13] and [14].

Proposition 2.4 $H(kA_\infty)$ coincides with its composition algebra.

Proof By theorem 1.2 $\{\mathrm{ind}_b^a\mid a\leqslant b,\ a,b\in\mathbb{N}\}$ are all indecomposable kA_∞-modules and $\{\mathrm{ind}_a^a\mid a\in\mathbb{N}\}$ are all simple kA_∞-modules. By corollary 2.1 we only need to prove that $[\mathrm{ind}_b^a]$, $a<b$, $a,b\in\mathbb{N}$ can be generated by $\{\mathrm{ind}_a^a\mid a\in\mathbb{N}\}$. By proposition 2.3 $[\mathrm{ind}_{b-1}^a]\circ[\mathrm{ind}_b^b]=[\mathrm{ind}_b^a]+[\mathrm{ind}_{b-1}^a\oplus\mathrm{ind}_b^b]$ and $[\mathrm{ind}_b^b]\circ[\mathrm{ind}_{b-1}^a]=[\mathrm{ind}_{b-1}^a\oplus\mathrm{ind}_b^b]$. So $[\mathrm{ind}_b^a]=[\mathrm{ind}_{b-1}^a]\circ[\mathrm{ind}_b^b]-[\mathrm{ind}_b^b]\circ[\mathrm{ind}_{b-1}^a]$. Then the assertion follows by induction on the length of modules. □

In the following we describe relations between $H(kA_\infty)$ and $H(kA_n)$ for $n\in\mathbb{N}$.

Theorem 2.3 Let $n\in\mathbb{N}$, then

$$\lim_{n\to+\infty}H(kA_n)=H(\lim_{n\to+\infty}kA_n)=H(kA_\infty).$$

Proof First we claim that $\lim\limits_{n\to+\infty}kA_n=kA_\infty$.

It's clear that kA_n is a subalgebra of kA_u if $n\leqslant u$. Denote by ϕ_u^n the inclusion map $kA_n\hookrightarrow kA_u$. It is obvious that $\phi_l^n=\phi_l{}^u\phi_u^n$ if $n\leqslant u\leqslant l$, $n,u,l\in\mathbb{N}$. It is clear that kA_n is a subalgebra of kA_∞ for any $n\in\mathbb{N}$, defining f_n to be the inclusion map $kA_n\hookrightarrow kA_\infty$. Certainly $f_n=f_u\phi_u^n$ if $n\leqslant u,n,u\in\mathbb{N}$.

To show that kA_∞ is the direct limit, we need to prove the following universal property: for any k- algebra X and a set of algebra morphisms $\{g_n: kA_n\to X\mid n\in\mathbb{N}\}$ such that $g_n=g_u\phi_u^n$ for $n\leqslant u,n,u\in\mathbb{N}$, there exists a unique algebra morphism σ from kA_∞ to X such that $g_n=\sigma f_n$ for any $n\in\mathbb{N}$.

Since the set of all finite length path elements of A_∞ is a basis of kA_∞, we define σ as $\sigma(p)=g_n(p)$, p is a path element of A_∞, $n=t(p)$. Let α,β be two paths in A_∞, set $n=t(\alpha)$, $u=t(\beta)$, then

$$\sigma(\beta\alpha)=g_u(\beta\alpha)=g_u(\beta)g_u(\alpha)=g_u(\beta)g_u\phi_u^n(\alpha)=g_u(\beta)g_n(\alpha)=\sigma(\beta)\sigma(\alpha),$$

which implies that σ is an algebra morphism.

Furthermore, for any path p in A_n we have $t(p)\leqslant n$, therefore

$$\sigma f_n(p)=\sigma(p)=g_{t(p)}(p)=g_n(p).$$

Moreover if such σ exists we must have $\sigma(p)=\sigma f_n(p)=g_n(p)$ for any finite path element p of A_∞ with $t(p)=n$. Now we have proved that $\lim\limits_{n\to+\infty}kA_n=kA_\infty$.

Secondly we shall prove that $\lim\limits_{n\to+\infty}H(kA_n)=H(kA_\infty)$.

By lemma 2.6 $H(kA_n)$ can be viewed as a subalgebra of $H(kA_u)$ if $n\leqslant u$, $n,u\in\mathbb{N}$ and $H(kA_n)$ can be viewed as a subalgebra of $H(kA_\infty)$ for $n\in\mathbb{N}$. Let $\psi_u^n: H(kA_n)\hookrightarrow H(kA_u)$ be the inclusion map, that is $\psi_u^n([E]) = [E]$ for any $E \in kA_n$-mod. Clearly $\psi_l^n = \psi_l^u \psi_u^n$ for any $n \leqslant u \leqslant l$. Define α_n to be the inclusion map $H(kA_n) \hookrightarrow H(kA_\infty)$ for $n \in \mathbb{N}$, that is $\alpha_n([E]) = [E]$ for any $E \in kA_n$-mod. Certainly we have $\alpha_n = \alpha_u \psi_u^n$ if $n \leqslant u$, $n, u \in \mathbb{N}$.

Again we need prove the following universal property: for any $\mathbb{Q}$-algebra X and a set of algebra morphisms $\{h_n : H(kA_n) \to X \mid n \in \mathbb{N}\}$ such that $h_n = h_u \psi_u^n$ for any $n \leqslant u$, there exists a unique algebra morphism $\theta : H(kA_\infty) \to X$ such that $h_n = \theta \alpha_n$ for any $n \in \mathbb{N}$.

Let E be a kA_∞-module of n-type, define $\theta([E]) = h_n([E])$, where E is viewed as a kA_n-module in the right hand side of the formula. Since the isoclass of all finite-dimensional kA_∞-modules forms a basis of $H(kA_\infty)$, θ can be uniquely spanned to a linear map from $H(kA_\infty)$ to X.

Let E, F be two finite-dimensional kA_∞-modules of n-type and u-type respectively. Let $v = \max\{n, u\}$, then E and F can also be viewed as kA_v-modules. Then by theorem 2.1 we have

$$\begin{aligned}\theta([E] \circ [F]) &= \theta\Big(\sum_{[L]} g_{E,F}^L [L]\Big) = \sum_{[L]} g_{E,F}^L \theta([L]) = \\ &\sum_{[L]} g_{E,F}^L h_v([L]) = h_v\Big(\sum_{[L]} g_{E,F}^L [L]\Big) = \\ &h_v([E] \circ [F]) = h_v([E]) h_v([F]) = \\ &h_v \psi_v{}^n([E]) h_v \psi_v^u([F]) = \\ &h_n([E]) h_u([F]) = \theta([E]) \theta([F]),\end{aligned}$$

where any kA_∞-module L appeared in the above summations is of v-type, and hence L becomes a kA_v-module. Thus θ is an algebra morphism.

Given any finite-dimensional kA_n-module E, then we can view E as a kA_∞-module, and there exist unique numbers $u \in \mathbb{N}$ with $u \leqslant n$ such that E is of u-type, therefore E also can be viewed as a kA_u-module. Since ψ_n^u is the inclusion map, we have

$$\theta \alpha_n([E]) = \theta([E]) = h_u([E]) = h_n \psi_n^u([E]) = h_n([E]),$$

and hence $\theta \alpha_n = h_n$.

Conversely, if such θ exists, we have $\theta([E]) = \theta \alpha_n([E]) = h_n([E])$ for any finite-dimensional kA_∞-module E of n-type. Now we have proved that $\lim\limits_{n \to +\infty} H(kA_n) = H(kA_\infty)$. □

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